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Non-Finite Dimensional Closed Vector Spaces of Universal Functions for Composition Operators*

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Let $H(\Omega)$ be the space of analytic functions on a complex region Ω, which is not the punctured plane. In this paper, we prove that if a sequence of automorphisms $\{\varphi_n\}_{n\geq 0}$ of Ω has the property that for every compact subset $K\subset\Omega$ there is a positive integer n such that $K\cap\varphi_n(K)=\varnothing$, then there exists an infinite dimensional closed vector subspace $F\subset H(\Omega)$ such that for all $f\in F\setminus\{0\}$ the orbit $\{f\circ\varphi_n\}_{n\geq 0}$ is dense in $H(\Omega)$. The corresponding result for the punctured plane is somewhat different and is also studied. © 1995 Academic Press, Inc.

1. INTRODUCTION AND TERMINOLOGY

Throughout this paper **C** will stand for the complex plane, Ω a region contained in **C**, **D** the open unit disk and $\mathbf{C}^\infty$ the extended complex plane and $\mathbf{C}^*=\mathbf{C}\setminus\{0\}$ the punctured plane. $H(\Omega)$ denotes, as usual, the space of holomorphic functions on Ω, endowed with the topology of uniform convergence on compact subsets.

If K is a compact subset of **C**, we denote by $A(K)$ the set of functions which are holomorphic in the interior of K and continuous on K. We denote by $\mathscr{K}_1(\Omega)$ the set of all compact subsets $K\subset\Omega$ whose complement is connected and by $\mathscr{K}(\Omega)$ the set of all compact subsets whose complement with respect to Ω has no connected, relatively compact components; in other words, the compact subsets which are Runge in Ω. It is obvious that $\mathscr{K}_1(\Omega)\subset\mathscr{K}(\Omega)$. If $S\subset\mathbf{C}$, then we call each connected component of $\mathbf{C}^\infty\setminus S$ a *hole*, including the connected component containing ∞. $Aut(\Omega)$ denotes the set of automorphisms of Ω and $\partial^\infty\Omega$ the boundary of Ω as

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a subset of $\mathbf{C}^{\infty}$. By l^2 we denote the Hilbert space of the sequences of complex numbers for which the norm

$$\|\{a_n\}_{n\geqslant 0}\|_2=\left(\sum_{n=0}^{\infty}|a_n|^2\right)^{1/2}$$

is finite. Finally, we denote by $L^2(T)$ the Hilbert space of the complex functions on the torus $T=\{z\in\mathbf{C}:|z|=1\}$ for which the norm

$$\|f\|_2=\left(\frac{1}{2\pi}\int_0^{2\pi}|f(e^{i\theta})|^2\,d\theta\right)^{1/2}$$

is finite. We shall make use of the isomorphism between these Hilbert spaces.

In 1929 G. D. Birkhoff [2] proved the following:

THEOREM. *There exists an entire function $f(z)$ such that to an arbitrary entire function $g(z)$ corresponds a sequence $\{a_n\}_{n\geqslant 0}$ depending on $g(z)$ and satisfying*

$$\lim_{n\to\infty} f(z+a_n)=g(z)$$

uniformly on any compact set.

Such a function is called *universal*. Since then, other authors have worked on this subject. In 1941 W. P. Seidel and J. L. Walsh [22] established an analogous theorem for the unit disk, replacing $z+a_n$ by $(z-a_n)/(z+\bar{a}_n)$ and letting $g(z)$ be holomorphic on a subregion of the unit disk.

In 1976 Luh [18] stated that given a sequence $\{a_n\}_{n\geqslant 0}$ with limit ∞, there exists an entire function f such that for every compact set K with connected complement in the complex plane and for every function $g\in A(K)$, there exists a subsequence $\{a_{n_k}\}_{k\geqslant 0}$ such that

$$\lim_{k\to\infty} f(z+a_{n_k})=g(z)$$

uniformly on K.

In 1989 Zappa [23] replaced the additive group of complex numbers $\mathbf{C}$ by the multiplicative group $\mathbf{C}^*$ and pointed out a generalization for a non-compact general Riemann surface S. He remarked the following: assume that the action of the group G of automorphisms of S is properly discontinuous, i.e., for every compact subset K of S there exists $\varphi\in G$ such that $K\cap\varphi(K)=\varnothing$; then under these conditions there exists a holomorphic function f on S such that for every compact subset K, with a fundamental system of simply connected neighborhoods, for every g holomorphic in the interior of K and continuous on its boundary, and for every $\varepsilon > 0$ there exists $\varphi \in G$ such that

$$\max_K |f \circ \varphi - g| \leqslant \varepsilon.$$

In 1984 Duios-Ruis [8] proved that in Birkhoff's theorem on translations, the universal vectors can have "arbitrarily slow growth." This result is refined further, and given an operator-theoretic twist, by Chan and Shapiro in [6]. Gethner and Shapiro furnished in [11] a single sufficient condition that provides a unified proof of universality in several situations, including theorems of Birkhoff, MacLane, Seidel and Walsh and many others. This same point of view is further advanced in the papers of Godefroy and Shapiro [12; Sections 4 and 5] and of Bourdon and Shapiro [5]. From the definition of universality it is derived that universal vectors form a residual set. See also [3], [13], [14], [19], and [20] for additional interesting results on universality, specially about derivative operators.

If $\{\varphi_n\}_{n \geqslant 0} \subset Aut(\Omega)$, then we may define their corresponding sequence of composition operators $T_n: H(\Omega) \to H(\Omega)$ $(n \geqslant 0)$ by $T_n(f) = f \circ \varphi_n$. Obviously, every T_n is a continuous linear operator on $H(\Omega)$. If $f \in H(\Omega)$, then f is said to be *universal* on $H(\Omega)$ (respectively on $A(K)$, where $K \subset \Omega$ is compact) if the orbit $\{T_n(f) = f \circ \varphi_n\}_{n \geqslant 0}$ is dense in $H(\Omega)$ ($A(K)$, respectively). It is clear that the above results can be expressed in these terms.

In [1] we introduced the following definition:

DEFINITION 1.1. Let $\{\varphi_n\}_{n \geqslant 0} \subset Aut(\Omega)$. We say that $\{\varphi_n\}_{n \geqslant 0}$ is *run-away* if for each compact subset $K \subset \Omega$ there exists a positive integer $n_0 = n_0(K)$ such that $K \cap \varphi_{n_0}(K) = \varnothing$.

In other words, the action of $\{\varphi_n\}_{n \geqslant 0}$ is properly discontinuous on Ω. The name *run-away* is introduced for the sake of brevity. It is an easy exercise to check that if ψ is an isomorphism from Ω onto Ω_1, then $\{\varphi_n\}_{n \geqslant 0}$ is run-away on Ω if and only if $\{\psi \circ \varphi_n \circ \psi^{-1}\}_{n \geqslant 0}$ is run-away on Ω_1. It is clear from the definition that if $\{K_n\}_{n \geqslant 0}$ is an exhaustive sequence of compact subsets in Ω, we only have to verify the condition on every K_n. In fact, we shall always assume, by extracting a subsequence of $\{\varphi_n\}_{n \geqslant 0}$ if necessary, that if $\{\varphi_n\}_{n \geqslant 0}$ is run-away on Ω and an exhaustive sequence of compact subsets $\{K_n\}_{n \geqslant 0}$ is given, then $K_n \cap \varphi_n(K_n) = \varnothing$. If it is so, then every subsequence of $\{\varphi_n\}_{n \geqslant 0}$ is also run-away.

In [1] we characterized the sequences $\{\varphi_n\}_{n \geqslant 0} \subset Aut(\Omega)$ where Ω is $\mathbf{C}$, $\mathbf{D}$ or $\mathbf{C}^*$ which are run-away and we proved that if we have a run-away sequence $\{\varphi_n\}_{n \geqslant 0} \subset Aut(\Omega)$, where Ω is not isomorphic to $\mathbf{C}^*$, then the set of universal functions on $H(\Omega)$ is a residual set of $H(\Omega)$ and, in addition, that the condition on $\{\varphi_n\}_{n\geq 0}$ of being run-away is necessary. In the case that Ω is isomorphic to $\mathbf{C}^*$ we only have that the set of the functions which are universal on $A(K)$ for all $K \in \mathscr{K}_1(\Omega)$ is a residual subset of $H(\Omega)$.

In a different setting it is well known that the set of nowhere diferentiable functions is a residual set in the space of the continuous functions. Recently, some authors have proved the existence of non-finite dimensional spaces of nowhere diferentiable functions, except the null function, of course (see [9], [15], and [21]). Since the set of universal functions is a residual set too, we ask for the existence of non-finite dimensional closed spaces of universal functions and give a positive answer. So, there are, not only topologically but also algebraically, a large number of universal functions. Our aim is to prove the following theorem:

THEOREM 1.2. *Let $\Omega \subset \mathbf{C}$ be a region, which is not isomorphic to $\mathbf{C}^*$. Let $\{\varphi_n\}_{n\geq 0} \subset Aut(\Omega)$ be a run-away sequence. Then there exists a non-finite dimensional closed vector subspace $F \subset H(\Omega)$ such that each $f \in F\backslash\{0\}$ is universal on $H(\Omega)$.*

This result complements a recent one of P. Bourdon [5], which states that if an operator T on a Banach space X has a hypercyclic (= universal) vector, then there is a dense, invariant subspace of X that consists, except for the zero vector, entirely of hypercyclic vectors. Special cases of Bourdon's result were proved by Godefroy and Shapiro in [12].

If Ω is a region of finite connectivity greater than 2, then $Aut(\Omega)$ is a finite set (see [16] for instance). So, there is not any sequence of automorphisms which can be run-away. If $\mathbf{C}^\infty\backslash\Omega$ has two connected components, then Ω may be isomorphic to the punctured unit disk, an annulus or $\mathbf{C}^*$ (see [17, pp. 68–69] for instance). It is easy to check that in the first and the second cases there is not any sequence of automorphisms which can be run-away either. So, apart from $\mathbf{C}^*$, the only interest of Definition 1.1 is in simply connected regions and regions with infinite connectivity.

2. PROOF OF THE MAIN RESULT

Some topological lemmas will be needed to prove Theorem 1.2. Definition 2.1, and Lemmas 2.2 and 2.3 can also be found in [1]. We repeat them here for the sake of completeness.

DEFINITION 2.1. Let $\Omega \subset \mathbf{C}$ be a region with infinity connectivity. We say that a connected component C of $\mathbf{C}^\infty\backslash\Omega$ is *isolated* if there exists an open set $U \subset \mathbf{C}^\infty$ such that $C \subset U$ and $U \cap C' = \emptyset$ for the remaining components C' of $\mathbf{C}^\infty \backslash \Omega$.

At this point it is convenient to point out that if $K \in \mathcal{K}(\Omega)$, then $\mathbf{C}^\infty \backslash K$ has finitely many connected components. Further, if $\mathbf{C}^\infty \backslash K$ has l connected components so has $\mathbf{C}^\infty \backslash \varphi(K)$, where φ is an automorphism of Ω.

LEMMA 2.2. *Let $\Omega \subset \mathbf{C}$ be a region of infinite connectivity and $\{\varphi_n\}_{n \geq 0} \subset Aut(\Omega)$ a run-away sequence. Then there exists a non-isolated, connected component C of $\mathbf{C}^\infty \backslash \Omega$, and a run-away subsequence $\{\varphi_{n_k}\}_{k \geq 0}$ such that for every compact set $K \subset \Omega$ and for every open set $U \subset \mathbf{C}^\infty$ with $C \subset U$, there exists a positive integer k_0 such that for every $k \geq k_0$ we have $\varphi_{n_k}(K) \subset U$.*

Proof. We may choose an exhaustive sequence $\{K_n\}_{n \geq 0}$ of connected compact subsets in Ω satisfying $\mathbf{C}^\infty \backslash K_n = \bigcup_{j \in J_n} U_j^n$, where the union is disjoint, J_n is a finite set and each U_j^n is an open subset of $\mathbf{C}^\infty$, in such a way that either U_j^n contains a unique, isolated, connected component of $\mathbf{C}^\infty \backslash \Omega$, or it contains a non-isolated, connected component. At this last case, it must contain infinitely many components of $\mathbf{C}^\infty \backslash \Omega$. We may suppose that $\mathbf{C}^\infty \backslash K_n$ has three or more components for each n. Since $\{\varphi_n\}_{n \geq 0}$ is run-away we have that, by extracting a subsequence, if necessary, $K_n \cap \varphi_n(K_n) = \emptyset$ for each n; hence, $\varphi_n(K_n) \subset \mathbf{C} \backslash K_n$ and, from the connection of $\varphi_n(K_n)$, there exists $j_0 \in J_n$ with $\varphi_n(K_n) \subset U_{j_0}^n$ where $U_{j_0}^n$ contains a non-isolated, connected component of $\mathbf{C}^\infty \backslash \Omega$, say, C_n. Hence, by the compactness of $\mathbf{C}^\infty$, there exists a connected component C of $\mathbf{C}^\infty \backslash \Omega$ and a subsequence $\{C_{n_k}\}_{k \geq 0}$ with the following property: given an open set $U \subset \mathbf{C}^\infty$ with $C \subset U$, there is a positive integer k_0 such that $C_{n_k} \subset U$ for all $k \geq k_0$. Then it is readily seen that C is non-isolated and $\{\varphi_{n_k}\}_{k \geq 0}$ satisfies the required conditions. Indeed, C is "approximated" by $\{C_{n_k}\}$ and, if $K \subset \Omega$ is compact and $U \subset \mathbf{C}^\infty$ is open with $C \subset U$, then $K \subset K_{n_k}$ for $k \geq k_0$. But the sets $U_{j_0}^{n_k}$ (j_0 certain index in J_{n_k}) can be made small containing C_{n_k}, so $U_{j_0}^{n_k} \subset U$ for k large enough. Thus $\varphi_{n_k}(K) \subset \varphi_{n_k}(K_{n_k}) \subset U_{j_0}^{n_k} \subset U$. ∎

In the sequel, if Ω has infinite connectivity and $\{\varphi_n\}_{n \geq 0} \subset Aut(\Omega)$ is a run-away sequence, we may assume that $\{\varphi_n\}_{n \geq 0}$ satisfies the property of the previous lemma by extracting a subsequence, if necessary.

LEMMA 2.3. *Let $\Omega \subset \mathbf{C}$ be a region of infinite connectivity, $\{\varphi_n\}_{n \geq 0} \subset Aut(\Omega)$ a run-away sequence and $K_1, K \in \mathcal{K}(\Omega)$. Then there exists a positive integer n_0 such that $K_1 \cap \varphi_{n_0}(K) = \emptyset$ and $K_1 \cup \varphi_{n_0}(K) \in \mathcal{K}(\Omega)$.*

Proof. Without loss of generality we may assume that K_1 and K are connected. Now, let l_1 and l be the number of holes of K_1 and K, respectively. Consider the connected component C furnished by Lemma 2.2. Then $U_1 = \mathbf{C}^\infty \setminus K_1$ is an open neighborhood of C. Since C is non-isolated there is an open neighborhood U and a connected component C_0 of $\mathbf{C}^\infty \setminus \Omega$ such that $C \subset U \subset U_1$ and $C_0 \subset U_1 \setminus U$. By Lemma 2.2, there exists a positive integer n_0 such that $\varphi_{n_0}(K) \subset U$. Clearly, $K_1 \cap \varphi_{n_0}(K) = \varnothing$. Then, the number of holes of $K_1 \cup \varphi_{n_0}(K)$ is $l_1 + l - 1$. It may happen that K_1 lies on a bounded connected component of $\mathbf{C}^\infty \setminus \varphi_{n_0}(K)$, or $\varphi_{n_0}(K)$ lies on a bounded connected component of $\mathbf{C}^\infty \setminus K_1$, or neither of both. In this last case there is nothing to prove. In the first two cases, $K_1 \cup \varphi_{n_0}(K)$ has, at least, $l_1 + l - 2$ holes that contain a hole of Ω. Let us suppose that $\varphi_{n_0}(K)$ lies on a bounded connected component of $\mathbf{C}^\infty \setminus K_1$ (the case that K_1 lies on a bounded component of $\mathbf{C}^\infty \setminus \varphi_{n_0}(K)$ can be handled analogously). We have to prove that there is a hole of Ω in the non-bounded connected component of $\mathbf{C}^\infty \setminus \varphi_{n_0}(K)$ which lies on the same hole of K_1 that $\varphi_{n_0}(K)$, since this shows that the complement of $K_1 \cup \varphi_{n_0}(K)$ has no relatively compact connected components. But, by the above construction, C_0 lies on the non-bounded connected component of $\mathbf{C}^\infty \setminus \varphi_{n_0}(K)$ and, consequently, it lies on the same hole of K_1 that $\varphi_{n_0}(K)$. ∎

In the remaining of this paper we may assume that the exhaustive sequence $\{K_n\}_{n \geq 0}$ is in $\mathcal{K}(\Omega)$. By using the previous lemma, it is easy to prove by induction the following lemma:

LEMMA 2.4. *Let $\Omega \subset \mathbf{C}$ be a region of infinite connectivity, $\{\varphi_n\}_{n \geq 0} \subset Aut(\Omega)$ a run-away sequence and $\{K_n\}_{n \geq 0}$ an exhaustive sequence of compact subsets of Ω. Then there exists a run-away subsequence $\{\varphi_{n_k}\}_{k \geq 0}$ and a subsequence of compact subsets $\{K_{n_k}\}_{k \geq 0}$ such that, for all finite set I of natural numbers with first element l and last element s, we have $K_{n_l} \cup (\bigcup_{i \in I} \varphi_{n_{k_i}}(K_{n_{k_i}}))$ is a disjoint union which belongs to $\mathcal{K}(\Omega)$ and is contained in $K_{n_{k_s}+1}$.*

The statement of the previous lemma is obvious if Ω is simply connected.

Proof of Theorem 1.2. Without loss of generality we may suppose that $\bar{\mathbf{D}} \subset \Omega$. Let $\{\varepsilon_m\}_{m \geq 0}$ be a sequence of positive numbers such that $\sum_{n=0}^{\infty} \varepsilon_m < 1$. Let $\{K_n\}_{n \geq 0}$ be an exhaustive sequence of compact subsets of Ω such that $\bar{\mathbf{D}} \subset K_0$. Actually, we consider that $\{K_n\}_{n \geq 0}$ and $\{\varphi_n\}_{n \geq 0}$ satisfy the same property as the subsequence obtained in Lemma 2.4. At last, let $\{p_n(z)\}_{n \geq 0}$ be a denumerable dense subset of $H(\Omega)$.

Since the proof has two very different parts, it will be convenient to divide it into two steps.

First step. By setting $i(m, n) = (n+m)(n+m+1)/2 + m$, we divide $\{\varphi_n\}_{n \geq 0}$ into infinitely many disjoint subsequences $\{\varphi_{i(m,n)}\}_{n \geq 0}$ $(m \geq 0)$, for each of them we will construct a corresponding function f_m such that:

(a) Each f_m is a universal function for $\{\varphi_{i(m,n)}\}_{n\geq 0}$, so is for $\{\varphi_n\}_{n\geq 0}$. In fact we have

$$\max_{K_n} |f_m(\varphi_{i(m,n)}(z)) - p_n(z)| < \frac{\varepsilon_m}{2^n} \qquad (n \geq 0).$$

(b) For $k \neq m$ the sequence $\{f_m(\varphi_{i(k,n)})\}_{n\geq 0}$ converges uniformly to zero on compact subsets. In fact, we have

$$\max_{K_n} |f_m(\varphi_{i(k,n)}(z))| < \frac{\varepsilon_m}{2^n} \qquad (n \geq 0).$$

(c) $\max_{\bar{\mathbf{D}}} |f_m(z) - z^m| < \varepsilon_m$.

Since $\{\varphi_n\}_{n\geq 0}$ and $\{K_n\}_{n\geq 0}$ satisfy the property of the subsequence obtained in Lemma 2.4, we have for each natural number $m \geq 0$ that the set $L_{m,0} = K_0 \cup (\bigcup_{j=0}^{i(m,0)} \varphi_j(K_j))$ is a disjoint union which is in $\mathscr{K}(\Omega)$ and it is contained in $K_i(m, 0) + 1$. Define on the compact subset $L_{m,0}$ the following function:

$$h_{m,0}(z) = \begin{cases} z^m, & \text{if } z \in K_0; \\ p_0(\varphi_{i(m,0)}^{-1}(z)), & \text{if } z \in \varphi_{i(m,0)}(K_{i(m,0)}); \\ 0, & \text{if } z \in \varphi_j(K_j) \text{ for } 0 \leq j < i(m,0). \end{cases}$$

Clearly, $h_{m,0}(z) \in A(L_{m,0})$. Hence, by Mergelyan's approximation theorem (see, for instance, [10, p. 119]), for each $m \geq 0$, there exists a rational function $q_{m,0} \in H(\Omega)$, with at most one pole in every connected component of $\mathbf{C}^\infty \backslash L_{m,0}$ (by the definition of $\mathscr{K}(\Omega)$) and no other poles, such that $\max_{L_{m,0}} |h_{m,0}(z) - q_{m,0}(z)| < \varepsilon_m / 2^{i(m,0)+1}$. So, we have

$$\max_{K_0} |q_{m,0}(z) - z^m| < \frac{\varepsilon_m}{2^{i(m,0)+1}},$$

$$\max_{\varphi_{i(m,0)}(K_{i(m,0)})} |q_{m,0}(z) - p_0(\varphi_{i(m,0)}^{-1}(z))| < \frac{\varepsilon_m}{2^{i(m,0)+1}},$$

$$\max_{\varphi_j(K_j)} |q_{m,0}(z)| < \frac{\varepsilon_m}{2^{i(m,0)+1}} \qquad (0 \leq j < i(m,0)).$$

Observe that the third maximum fails to appear when $m = 0$.

By induction, we may define, for each $m \geq 0$, the set $L_{m,n} = K_{i(m,n-1)+1} \cup (\bigcup_{j=i(m,n-1)+1}^{i(m,n)} \varphi_j(K_j))$ and again we have that it is a disjoint union which is in $\mathscr{K}(\Omega)$ and it is contained in $K_{i(m,n)+1}$. Define on the compact subset $L_{m,n}$ the following function:

$$h_{m,n}(z)=\begin{cases} q_{m,n-1}(z), & \text{if } z\in K_{i(m,n-1)+1};\\ p_n(\varphi^{-1}_{i(m,n)}(z)), & \text{if } z\in \varphi_{i(m,n)}(K_{i(m,n)});\\ 0, & \text{if } z\in\varphi_j(K_j)\ (i(m,n-1)<j<i(m,n)).\end{cases}$$

Clearly, $h_{m,n}(z)\in A(L_{m,n})$ for every $m\geqslant 0$. Hence, by Mergelyan's approximation theorem again, for each $m\geqslant 0$, there exists a rational function $q_{m,n}\in H(\Omega)$, with at most one pole in every connected component of $\mathbf{C}^{\infty}\backslash L_{m,n}$ (by the definition of $\mathscr{K}(\Omega)$) and no other poles, such that $\max_{z\in L_{m,n}}|h_{m,n}(z)-q_{m,n}(z)|<\varepsilon_m/2^{i(m,n)+1}$. So, we have

$$\max_{K_{i(m,n-1)+1}}|q_{m,n}(z)-q_{m,n-1}(z)|<\frac{\varepsilon_m}{2^{i(m,n)+1}},$$

$$\max_{\varphi_{i(m,n)}(K_{i(m,n)})}|q_{m,n}(z)-p_n(\varphi^{-1}_{i(m,n)}(z))|<\frac{\varepsilon_m}{2^{i(m,n)+1}},$$

$$\max_{\varphi_j(K_j)}|q_{m,n}(z)|<\frac{\varepsilon_m}{2^{i(m,n)+1}}\qquad (i(m,n-1)<j<i(m,n)).$$

Plainly, $\{q_{m,n}\}_{n\geqslant 0}$ converges uniformly on compact subsets of Ω to a function $f_m\in H(\Omega)$ for every $m\geqslant 0$. These functions may be written, for any $n\geqslant 0$, as

$$f_m(z)=q_{m,n}(z)+\sum_{k=n}^{\infty}(q_{m,k+1}(z)-q_{m,k}(z)).$$

Hence, since $\bar{\mathbf{D}}\subset K_0\subset K_{i(m,k)+1}$ for all $k\geqslant 0$ we have

$$\begin{aligned}\max_{\bar{\mathbf{D}}}|f_m(z)-z^m| &\leqslant \max_{K_0}|q_{m,0}(z)-z^m|+\sum_{k=0}^{\infty}\max_{K_{i(m,k)+1}}|q_{m,k+1}(z)-q_{m,k}(z)|\\ &<\sum_{k=0}^{\infty}\frac{\varepsilon_m}{2^{i(m,k)+1}}<\varepsilon_m,\end{aligned}$$

for every $m\geqslant 0$. Therefore we have (c).

To see, for each $m\geqslant 0$, that f_m is universal for $\{\varphi_{i(m,n)}\}_{n\geqslant 0}$, it suffices to observe that for $z\in\varphi_{i(m,n)}(K_{i(m,n)})\subset K_{i(m,n)+1}$ we have

$$\begin{aligned}&|f_m(z)-p_n(\varphi^{-1}_{i(m,n)}(z))|\\ &\quad\leqslant |q_{m,n}(z)-p_n(\varphi^{-1}_{i(m,n)}(z))|+\sum_{k=n}^{\infty}|q_{m,k+1}(z)-q_{n,k}(z)|\\ &\quad<\sum_{k=n}^{\infty}\frac{\varepsilon_m}{2^{i(m,k)+1}}<\frac{\varepsilon_m}{2^{i(m,n)}}.\end{aligned}$$

So, we have

$$\begin{aligned}
&\max_{K_n} |f_m(\varphi_{i(m,n)}(z)) - p_n(z)| \\
&\quad \leqslant \max_{K_{i(m,n)}} |f_m(\varphi_{i(m,n)}(z)) - p_n(z)| \\
&\quad = \max_{\varphi_{i(m,n)}(K_{i(m,n)})} |f_m(z) - p_n(\varphi_{i(m,n)}^{-1}(z))| \\
&\quad < \frac{\varepsilon_m}{2^{i(m,n)}} < \frac{\varepsilon_m}{2^n}.
\end{aligned}$$

Hence, for every $m \geqslant 0$

$$\lim_{n \to \infty} |f_m(\varphi_{i(m,n)}(z)) - p_n(z)| = 0$$

uniformly on compact subsets.

Since, for every $m \geqslant 0$, the sequence $\{f_m(\varphi_{i(m,n)})\}_{n \geqslant 0}$ is near enough to $\{p_n\}_{n \geqslant 0}$ and we can extract from the latter a subsequence converging uniformly on compact subsets to any $f \in H(\Omega)$ we have that we can extract from the former a subsequence converging uniformly on compact subsets to any $f \in H(\Omega)$. Which proves the universality of f_m for every $m \geqslant 0$. This is (a).

We are ready to prove that $\{f_m(\varphi_{i(k,n)})\}_{n \geqslant 0}$ converges uniformly on compact subsets to the null function for $k \neq m$. Fix $n \in \{0, 1, 2, ...\}$ and denote by r the unique natural number such that $i(m, r-1) < i(k, n) < i(m, r)$. Since $K_n \subset K_{i(k,n)}$ and $\varphi_{i(k,n)}(K_{i(k,n)}) \subset K_{i(k,n)+1} \subset K_{i(m,r)+1}$, the following inequalities hold:

$$\begin{aligned}
&\max_{K_n} |f_m(\varphi_{i(k,n)}(z))| \\
&\quad \leqslant \max_{K_{i(k,n)}} |f_m(\varphi_{i(k,n)}(z)| \\
&\quad \leqslant \max_{K_{i(k,n)}} |q_{m,r}(\varphi_{i(k,n)}(z))| + \sum_{l=r}^{\infty} \max_{K_{i(m,l)+1}} |q_{m,l+1}(z) - q_{m,l}(z)| \\
&\quad < \sum_{l=r}^{\infty} \frac{\varepsilon_m}{2^{i(m,l)+1}} < \frac{\varepsilon_m}{2^{i(m,r)}} < \frac{\varepsilon_m}{2^{i(k,n)}} < \frac{\varepsilon_m}{2^n}.
\end{aligned}$$

This is (b). So, (a), (b), and (c) are fulfilled.

Second step. Let E be the vector space consisting of all the series $\sum_{m=0}^{\infty} \alpha_m f_m$ which converge uniformly on compact subsets of Ω and F the closure of E in $H(\Omega)$. Clearly, F is closed. So, we have only to prove that it is a non-finite dimensional vector space of universal functions.

First, we prove that $\{f_m\}_{m \geqslant 0}$ is a basic sequence on $L^2(T)$, so they are linearly independent on $H(\Omega)$. Let $\{z_m^*\}_{m \geqslant 0}$ be the coefficient functionals corresponding to the basic sequence $\{z^m\}_{m \geqslant 0}$. By using (c) and the fact that $\|z_m^*\|_2 = 1$ for all m, we have on $L^2(T)$

$$\sum_{m=0}^{\infty} \|z_m^*\|_2 \, \|z^m - f_m\|_2 \leqslant \sum_{m=0}^{\infty} \max_{\bar{D}} |z^m - f_m| < \sum_{m=0}^{\infty} \varepsilon_m < 1.$$

As $\{z^m\}_{m \geqslant 0}$ is a basic sequence on $L^2(T)$, we have that $\{f_m\}_{m \geqslant 0}$ is a basic sequence equivalent on $L^2(T)$ to $\{z^m\}_{m \geqslant 0}$ (see [7, Theorem 9, p. 46]). This means that the closed linear span in $L^2(T)$ generated by $\{z^m\}_{m \geqslant 0}$ is isomorphic to the closed linear span in $L^2(T)$ generated by $\{f_m\}_{m \geqslant 0}$. So, we can associate to each element of F a unique sequence $\{\alpha_m\}_{m \geqslant 0}$ which is in l^2. This may be made in the following way. If $f \in F$, then there is a sequence of series of E which converges to f. By the continuity of $\|\cdot\|_2$ with respect to the maximum norm we have that this sequence of series converges to f on $L^2(T)$. Thus f has a representation as a series on $L^2(T)$. Of course, this series may not converge uniformly on compact subsets. Moreover, by using the maximum modulus principle, it is easy to see that the only function that we associate the null sequence is the null function.

Now we prove that for each series $\sum_{m=0}^{\infty} \alpha_m f_m$ converging uniformly on compact subsets, which is not the null function, is a universal function for $\{\varphi_n\}_{n \geqslant 0}$. Since $\sum_{m=0}^{\infty} \alpha_m f_m$ is not the null function there is an $\alpha_k \neq 0$. Since every non-zero scalar multiple of a universal function is again universal, we may suppose that $\alpha_k = 1$. To see that $\sum_{m=0}^{\infty} \alpha_m f_m$ is universal for $\{\varphi_n\}_{n \geqslant 0}$ we have only to check that

$$\lim_{n \to \infty} \left(\sum_{m=0}^{\infty} \alpha_m f_m(\varphi_{i(k,n)}(z)) - p_n(z) \right) = 0 \tag{1}$$

uniformly on compact subsets. This is readily seen by computing

$$\max_{K_n} \left| \sum_{0}^{\infty} \alpha_m f_m(\varphi_{i(k,n)}(z)) - p_n(z)) \right|. \tag{2}$$

By triangle inequality (2) is less than

$$\max_{K_n} |f_k(\varphi_{i(k,n)}(z)) - p_n(z)| + \max_{K_n} \sum_{m \neq k} |\alpha_m f_m(\varphi_{i(k,n)}(z)|. \tag{3}$$

By using (a) and (b) we have that (3) is less than

$$\frac{\varepsilon_k}{2^n} + \sum_{m \neq k} |\alpha_m| \frac{\varepsilon_m}{2^n} = \frac{1}{2^n} \sum_{m=0}^{\infty} |\alpha_m| \, \varepsilon_m. \tag{4}$$

and by the Cauchy–Schwarz inequality we obtain that (4) is less than

$$\frac{\|\{\alpha_m\}_{m\geq 0}\|_2 \, \|\{\varepsilon_m\}_{m\geq 0}\|_2}{2^n} < \frac{\|\{\alpha_m\}_{m\geq 0}\|_2}{2^n},$$

which tends to 0 when n tends to ∞, so we have (1).

It remains to show that for all $f \in \bar{E}$ except the null function, f is a universal function for $\{\varphi_n\}_{n\geq 0}$. Let $\sum_{m=0}^{\infty} \alpha_m f_m$ be its representation on $L^2(T)$. We suppose again that there is an $\alpha_k = 1$. Let $\{\sum_{m=0}^{\infty} \alpha_m^l f_m\}_{l\geq 0}$ be a sequence of series of E converging on $H(\Omega)$ to f. It is obvious that we may consider that $\alpha_k^l = 1$ for all $l \in \mathbf{N}$. Analogously as before, to see that f is a universal function, it suffices to verify that

$$\lim_{n\to\infty} (f(\varphi_{i(k,n)}(z)) - p_n(z)) = 0 \tag{5}$$

uniformly on compact subsets. For this, fix l and n and estimate

$$\begin{aligned}
&\max_{K_n} |f(\varphi_{i(k,n)}(z)) - p_n(z)| \\
&\quad \leq \max_{K_n} \left| f(\varphi_{i(k,n)}(z)) - \sum_{m=0}^{\infty} \alpha_m^l f_m(\varphi_{i(k,n)}(z)) \right| \\
&\quad\quad + \max_{K_n} \left| \sum_{m=0}^{\infty} \alpha_m^l f_m(\varphi_{i(k,n)}(z)) - p_n(z) \right| \\
&\quad < \max_{K_n} \left| f(\varphi_{i(k,n)}(z)) - \sum_{m=0}^{\infty} \alpha_m^l f_m(\varphi_{i(k,n)}(z)) \right| \\
&\quad\quad + \frac{\|\{\alpha_m^l\}_{m\geq 0}\|_2}{2^n}.
\end{aligned} \tag{6}$$

As the sequence of series converges to f and $\|\{\alpha_m^l\}_{m\geq 0}\|_2$ does it to $\|\{\alpha_m\}_{m\geq 0}\|_2$ when l tends to ∞, we have that there is a $l(n)$ such that (6) is less than $1/2^n + (\|\{\alpha_m\}_{m\geq 0}\|_2 + 1)/2^n$, which tends to 0. So, we have (5). This ends the second step and the proof. ∎

3. The Case $\mathbf{C}^*$

If $\Omega = \mathbf{C}^*$ we find some differences. We have no function that can be universal on $H(\mathbf{C}^*)$ (see [1] and [14]). But given a run-away sequence $\{\varphi_n\}_{n\geq 0}$ in $\mathbf{C}^*$ there are universal functions for $\{\varphi_n\}_{n\geq 0}$ on $A(K)$ for all $K \in \mathscr{K}_1(\mathbf{C}^*)$. As a matter of fact, we also have that the set of such functions is a residual set. Moreover, it is possible to construct a closed non-finite vector space of universal functions on $A(K)$ for all $K \in \mathscr{K}_1(\mathbf{C}^*)$. More precisely, we have the following theorem:

THEOREM 3.1. *Let $\Omega \subset \mathbf{C}$ be any complex region. Let $\{\varphi_n\}_{n \geq 0} \subset Aut(\Omega)$ be a run-away sequence. Then there exists a non-finite dimensional closed vector subspace $F \subset H(\Omega)$ of functions such that every $f \in F \backslash \{0\}$ is universal on $A(K)$ for all $K \in \mathscr{K}_1(\Omega)$.*

To prove this theorem we need the following lemma:

LEMMA 3.2. *For every region $\Omega \subset \mathbf{C}$ there exists a sequence $\{K_n\}_{n \geq 0} \subset \mathscr{K}_1(\Omega)$ such that for every $K \in \mathscr{K}_1(\Omega)$ there is a positive integer n_0 such that $K \subset K_{n_0}$.*

Proof. Consider the denumerable set $\{U_n\}_{n \geq 0}$ of all connected, finite unions of chordal balls with rational centers and rational radii, which contain the compact set $L_1 = \mathbf{C}^\infty \backslash \Omega$. Define $K_n = \mathbf{C}^\infty \backslash U_n$. If $K \in \mathscr{K}_1(\Omega)$, then we may construct a connected compact set L of $\mathbf{C}^\infty$ with $L \cap K = \varnothing$ and $L_1 \subset L$. Let r be a positive rational number with the chordal distance between K and L greater than r. Cover L by chordal balls of radius r with rational centers such that the intersection of each of these balls with L is not empty. We may extract a finite covering by such balls. Denote by U the union of these balls. It is obvious that K is contained in $\mathbf{C}^\infty \backslash U$ and that $U = U_n$ for some positive integer n, which ends the proof. ∎

Proof of Theorem 3.1. The proof is rather the same as that of Theorem 1.2 but with a few modifications. We suppose again that $\bar{\mathbf{D}} \subset \Omega$. Let $\{\varepsilon_m\}_{m \geq 0}$ be a sequence of positive numbers such that $\sum_{n=0}^{\infty} \varepsilon_m < 1$. We also consider a denumerable dense subset of $H(\Omega)$, $\{p_n(z)\}_{n \geq 0}$, and we consider the sequence of compact subsets $\{K'_n\}_{n \geq 0}$ given by Lemma 3.2. Note that it is suficient to prove the theorem for $A(K'_n)$ for all n.

By using that $\{\varphi_n\}_{n \geq 0}$ is run-away we choose an exhaustive sequence of compact subsets of Ω, $\{K_n\}_{n \geq 0} \subset \mathscr{K}(\Omega)$, such that for a subsequence of $\{\varphi_n\}_{n \geq 0}$ that we re-enumerate by $\{\varphi_{t,n} : 0 \leq t \leq n\}$ we have that the compact subsets $L_n = K_n \cup (\bigcup_{t=0}^n \varphi_{t,n}(K'_t))$ is a disjoint union which is contained in K_{n+1} and it is in $\mathscr{K}(\Omega)$ for every natural number $n \geq 0$. We also assume that $\bar{\mathbf{D}} \subset K_0$. It is clear that L_n is in $\mathscr{K}(\Omega)$ for all n. We also divide the proof into two steps.

First step. We again put $i(m,n) = (n+m)(n+m+1)/2 + m$ and we obtain, for each $t \geq 0$ and $m \geq 0$, a subsequence $\{\varphi_{t,i(m,n)} : n \geq 0;\ i(m,n) \geq t\}$ from $\{\varphi_{t,n}\}_{n \geq t}$. From these subsequences, by a way of double induction, we will construct a sequence of functions $f_m \in H(\Omega)$ $(m \geq 0)$ in such a way that:

(a) For every $m \geqslant 0$ and each $t \geqslant 0$, f_m is a universal function for $\{\varphi_{t,\, i(m,n)} : i(m, n) \geqslant t\}$ on $A(K'_t)$. So is for $\{\varphi_n\}_{n \geqslant 0}$ on $A(K'_t)$ for all natural number $t \geqslant 0$. In fact we have

$$\max_{K'_t} |f_m(\varphi_{t,\, i(m,n)}(z)) - p_n(z)| < \frac{\varepsilon_m}{2^n} \qquad (n \geqslant 0;\, i(m, n) \geqslant t).$$

(b) For every $m \geqslant 0$ and for each $t \geqslant 0$, if $k \neq m$ the sequence $\{f_m(\varphi_{t,\, i(k,n)}) : i(k, n) \geqslant t\}$ converges uniformly to zero on K'_t. In fact, we have for $n \geqslant n(t)$:

$$\max_{K'_t} |f_m(\varphi_{t,\, i(k,n)}(z))| < \frac{\varepsilon_m}{2^n} \qquad (n \geqslant 0;\, i(k, n) \geqslant t).$$

(c) $\max_{\bar{\mathbf{D}}} |f_m(z) - z^m| < \varepsilon_m$.

We consider $L_{m,0} = K_0 \cup (\bigcup_{j=0}^{i(m,0)} (\bigcup_{t=0}^{j} \varphi_{t,j}(K'_t)))$ which are in $\mathscr{K}(\Omega)$ and are contained in $K_{i(m,0)+1}$, respectively. Define on each compact subset $L_{m,0}$ the following function:

$$h_{m,0}(z) = \begin{cases} z^m, & \text{if } z \in K_0; \\ p_0(\varphi_{t,\, i(m,0)}^{-1}(z)), & \text{if } z \in \varphi_{t,\, i(m,0)}(K'_t) \quad (0 \leqslant t \leqslant i(m, 0)); \\ 0, & \text{if } z \in \varphi_{t,j}(K'_t) \quad (0 \leqslant t \leqslant j < i(m, 0)). \end{cases}$$

Clearly, $h_{m,0}(z) \in A(L_{m,0})$. Then, for each m, by Mergelyan's approximation theorem there exists a rational function $q_{m,0} \in H(\Omega)$, with at most one pole in every connected component of $\mathbf{C}^{\infty} \backslash L_{m,0}$ and no other poles, such that $\max_{L_{m,0}} |h_{m,0}(z) - q_{m,0}(z)| < \varepsilon_m / 2^{i(m,0)+1}$. So, we have

$$\max_{K_0} |q_{m,0}(z) - z^m| < \frac{\varepsilon_m}{2^{i(m,0)+1}},$$

$$\max_{\varphi_{t,\, i(m,0)}(K'_t)} |q_{m,0}(z) - p_0(\varphi_{t,\, i(m,0)}^{-1}(z))| < \frac{\varepsilon_m}{2^{i(m,0)+1}} \qquad (0 \leqslant t \leqslant i(m, 0)),$$

$$\max_{\varphi_{t,j}(K'_t)} |q_{m,0}(z)| < \frac{\varepsilon_m}{2^{i(m,0)+1}} \qquad (0 \leqslant t \leqslant j < i(m, 0)).$$

By induction, for any n, we have that

$$L_{m,n} = K_{i(m,n-1)+1} \cup \left(\bigcup_{j=i(m,n-1)+1}^{i(m,n)} \left(\bigcup_{t=0}^{j} \varphi_{t,j}(K'_t) \right) \right)$$

is a compact subset which is contained in $K_{i(m,n)+1}$ and belongs to $\mathcal{K}(\Omega)$. Define, for each $m \geqslant 0$, on $L_{m,n}$ the following function:

$$h_{m,n}(z)=\begin{cases} q_{m,n-1}(z), & z \in K_{i(m,n-1)+1};\\ p_n(\varphi_{t,i(m,n)}^{-1}(z)), & z \in \varphi_{t,i(m,n)}(K'_t) \quad (0 \leqslant t \leqslant i(m,n));\\ 0, & z \in \varphi_{t,j}(K'_t) \quad (i(m,n-1)<j<i(m,0);\\ & \qquad 0 \leqslant t \leqslant j). \end{cases}$$

Clearly, for each $m \geqslant 0$, we obtain that $h_{m,n}(z) \in A(L_{m,n})$. Then, for each m, by Mergelyan's approximation theorem there exists a rational function $q_{m,n} \in H(\Omega)$, with at most one pole in every connected component of $\mathbf{C}^{\infty} \backslash L_{m,n}$ and no other poles, such that $\max_{L_{m,n}} |h_{m,n}(z)-q_{m,n}(z)|<\varepsilon_m/2^{i(m,n)+1}$. So, we have

$$\max_{K_{i(m,n-1)+1}} |q_{m,n}(z)-q_{m,n-1}(z)|<\frac{\varepsilon_m}{2^{i(m,n)+1}},$$

$$\max_{\varphi_{t,i(m,n)}(K'_t)} |q_{m,n}(z)-p_n(\varphi_{t,i(m,n)}^{-1}(z))|<\frac{\varepsilon_m}{2^{i(m,n)+1}} \quad (0 \leqslant t \leqslant i(m,n)),$$

$$\max_{\varphi_{t,j}(K'_t)} |q_{m,n}(z)|<\frac{\varepsilon_m}{2^{i(m,n)+1}} \quad (i(m,n-1)<j<i(m,n);\, 0 \leqslant t \leqslant j).$$

It is clear, for each $m \geqslant 0$, that $\{q_{m,n}(z)\}_{n \geqslant 0}$ converges uniformly on compact subsets of Ω to a function $f_m \in H(\Omega)$. These functions may be written, for any $n \geqslant 0$, as

$$f_m(z)=q_{m,n}(z)+\sum_{k=n}^{\infty}(q_{m,k+1}(z)-q_{m,k}(z)).$$

Hence, analogously as in Theorem 1.2 we may obtain (c).

To see that f_m is universal on $A(K'_t)$ for $\{\varphi_{t,i(m,n)}: i(m,n) \geqslant t\}$ for all $t \geqslant 0$ we compute, for $z \in \varphi_{t,i(m,n)}(K'_t) \subset K_{i(m,n)+1}$ the following:

$$\begin{aligned} &|f_m(z)-p_n(\varphi_{t,i(m,n)}^{-1}(z))|\\ &\quad \leqslant |q_{m,n}(z)-p_n(\varphi_{t,i(m,n)}^{-1}(z))|+\sum_{k=n}^{\infty}|q_{m,k+1}(z)-q_{m,k}(z)|\\ &\quad <\sum_{k=n}^{\infty}\frac{\varepsilon_m}{2^{i(m,k)+1}}<\frac{\varepsilon_m}{2^{i(m,n)}}<\frac{\varepsilon_m}{2^n}. \end{aligned}$$

So we have, for all $t \geqslant 0$,

$$\max_{K'_t} |f_m(\varphi_{t,\, i(m,n)}(z)) - p_n(z)|$$

$$= \max_{\varphi_{t,\, i(m,n)}(K'_t)} |f_m(z) - p_n(\varphi^{-1}_{t,\, i(m,n)}(z))| < \frac{\varepsilon_m}{2^n}.$$

Hence, for all $t \geqslant 0$

$$\lim_{n \to \infty} (f_m(\varphi_{t,\, i(m,n)}(z)) - p_n(z)) = 0$$

uniformly on K'_t. Let $n(m, t)$ be the first natural number for which $i(m, n) \geqslant t$. Since by Mergelyan's approximation theorem $\{p_n\}_{n \geqslant 0}$ is dense in $A(K'_t)$ we have that $\{p_n\}_{n \geqslant n(m,t)}$ is dense in $A(K'_t)$ for all natural number $t \geqslant 0$. So we have proved the universality of f_m on $A(K'_t)$ for all $t \geqslant 0$. This is (a).

It remains to prove that, for fixed $m \geqslant 0$, $t \geqslant 0$ and $k \geqslant 0$ with $k \neq m$, the sequence $\{f_m(\varphi_{t,\, i(k,n)}) : n \geqslant 0;\ i(k, n) \geqslant t\}$ converges uniformly on compact subsets to zero on K'_t. Note that, for $n \geqslant n(t)$, $K'_t \subset K_{i(0,n)} \subset K_{i(k,n)}$ and $\varphi_{t,\, i(k,n)}(K_{i(k,n)}) \subset K_{i(k,n)+1}$. If r is the unique natural number with $i(m, r-1) + 1 \leqslant i(k, n) < i(m, r)$, we estimate

$$\max_{K'_t} |f_m(\varphi_{t,\, i(k,n)}(z))|$$

$$\leqslant \max_{K_{i(k,n)}} |q_{m,n}(\varphi_{t,\, i(k,n)}(z))| + \sum_{l=r}^{\infty} \max_{K_{i(m,l)+1}} |q_{m,l+1}(z) - q_{m,l}(z)|$$

$$\leqslant \sum_{l=r}^{\infty} \frac{\varepsilon_m}{2^{i(m,l)+1}} < \frac{\varepsilon_m}{2^{i(m,r)}} < \frac{\varepsilon_m}{2^{i(k,n)}} < \frac{\varepsilon_m}{2^n}.$$

So (b) and the first step are completed.

Second step. All this step is analogous to that of Theorem 1.2. So, we define E and F as before. To see that $\sum_{m=0}^{\infty} \alpha_m f_m$ is universal for $\{\varphi_n\}_{n \geqslant 0}$, where $\alpha_k = 1$, we have only to check that

$$\lim_{n \to \infty} \left(\sum_{m=0}^{\infty} \alpha_m f_m(\varphi_{t,\, i(k,n)}(z)) - p_n(z) \right) = 0 \tag{7}$$

uniformly K'_t for all $t \geqslant 0$. Again, we can see it by computing for n large enough

$$\max_{K'_t} \left| \sum_0^{\infty} \alpha_m f_m(\varphi_{t,\, i(k,n)}(z)) - p_n(z)) \right|,$$

which handled as before is less than $\|\{\alpha_m\}_{m\geqslant 0}\|_2/2^n$. This tends to 0 when n tends to ∞ so we have (7).

Let $\{\sum_{m=0}^{\infty}\alpha_m^l f_m\}_{l\geqslant 0}$ be a sequence of series of E converging on $H(\Omega)$ to f. This implies the sequence of series conveges uniformly on K'_t for all $t\geqslant 0$. We also consider that $\alpha_k^l=1$ for all $l\in\mathbf{N}$. Analogously as before, to see that f is a universal function, it suffices to verify that

$$\lim_{n\to\infty}(f(\varphi_{t,\,i(k,n)}(z))-p_n(z))=0 \tag{8}$$

uniformly on K'_t for all $t\geqslant 0$. For this, fix l and n large enough and estimate

$$\begin{aligned}
&\max_{K'_t}|f(\varphi_{t,\,i(k,n)}(z))-p_n(z)|\\
&\quad\leqslant \max_{K'_t}\left|f(\varphi_{t,\,i(k,n)}(z))-\sum_{m=0}^{\infty}\alpha_m^l f_m(\varphi_{t,\,i(k,n)}(z))\right|\\
&\qquad+\max_{K'_t}\left|\sum_{m=0}^{\infty}\alpha_m^l f_m(\varphi_{t,\,i(k,n)}(z))-p_n(z)\right|\\
&\quad<\max_{K'_t}\left|f(\varphi_{t,\,i(k,n)}(z))-\sum_{m=0}^{\infty}\alpha_m^l f_m(\varphi_{t,\,i(k,n)}(z))\right|\\
&\qquad+\frac{\|\{\alpha_m^l\}_{m\geqslant 0}\|_2}{2^n}.
\end{aligned}\tag{9}$$

As the sequence of series converges to f and $\|\{\alpha_m^l\}_{m\geqslant 0}\|_2$ does it to $\|\{\alpha_m\}_{m\geqslant 0}\|_2$ when l tends to ∞, we have that there is a $l(n)$ such that (9) is less than $1/2^n+(\|\{\alpha_m\}_{m\geqslant 0}\|_2+1)/2^n$, which tends to 0. So, we have (8). This ends the second step and the proof. ∎

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REFERENCES

1. L. BERNAL AND A. MONTES, Universal functions for composition operators, *Complex Variables* **27** (1995), 47–56.
2. G. D. BIRKHOFF, Démonstration d'un théorème élémentaire sur les fonctions entières, *C. R. Acad. Sci. Paris* **189** (1929), 473–475.
3. C. BLAIR AND L. A. RUBEL, A triply universal entire function, *Enseign. Math.* **30** (1984), 269–274.
4. P. BOURDON, Invariant manifolds of hypercyclic vectors, *Proc. Amer. Math. Soc.* **118** (1993), 845–847.

5. P. BOURDON AND J. H. SHAPIRO, Cyclic composition operators on H^2, *Proc. Symp. Pure Math.* **51**, Part 2 (1990), 45–53.
6. K. C. CHAN AND J. H. SHAPIRO, The cyclic behavior of translation operators on Hilbert spaces of entire functions, *Indiana Univ. Math. J.* **40** (1991), 1421–1449.
7. J. DIESTEL, "Sequences and Series in Banach Spaces," Springer-Verlag, New York, 1984.
8. S. M. DUIOS-RUIS, Universal functions of the structure of the space of entire functions, *Soviet Math. Dokl.* **30**(3) (1984), 713–716.
9. V. P. FONF, V. I. GURARII, AND M. J. KADEC, On certain subspace of $\mathscr{C}$, preprint.
10. D. GAIER, "Lectures on Complex Appoximation," Birkhäuser, Boston, 1987.
11. R. M. GETHNER AND J. H. SHAPIRO, Universal vectors for operators on spaces of holomorphic functions, *Proc. Amer. Math. Soc.* **100**(2) (1987), 281–288.
12. G. GODEFROY AND J. H. SHAPIRO, Operators with dense, invariant cyclic vector manifolds, *J. Funct. Anal.* **98** (1991), 229–269.
13. K. G. GROßE-ERDMANN, Holomorphe Monster und universelle Funktionen, *Mitt. Math. Sem. Giessen* **176** (1987), 1–84.
14. K. G. GROßE-ERDMANN, On the universal functions of G. R. MacLane, *Complex Variables* **15** (1990), 193–196.
15. V. I. GURARII, On subspaces of $\mathscr{C}$ consisting of non-differentiable functions (in Russian), *C. R. Acad. Bulgare Sci.* **74**(5) (1991), 13–16.
16. M. HEINS, On the number of 1-1 directly conformal maps which a multiply-connected plane region of finite connectivity p (>2) admits onto itself, *Bull. Amer. Math. Soc.* **52** (1946), 454–457.
17. G. JULIA, "Leçons sur la représentation conforme des aires multiplement connexes," Gauthier-Villars, Paris, 1939.
18. W. LUH, On universal functions, *Colloq. Math. Soc. János Bolyai* **19** (1976), 503–511.
19. W. LUH, Approximation by antiderivatives, *Constr. Approx.* **2** (1986), 179–187.
20. W. LUH, Holomorphic Monsters, *J. Approx. Theory* **53** (1988), 128–144.
21. L. RODRÍGUEZ-PIAZZA, Every separable Banach space is isometric to a space of continuous nowhere differentiable functions, *Proc. Amer. Math. Soc.*, to appear.
22. W. P. SEIDEL AND J. L. WALSH, On approximation by Euclidean and non-Euclidean translates of an analytic function, *Bull. Amer. Math. Soc.* **47** (1941), 916–920.
23. P. ZAPPA, On universal holomorphic functions, *Bollettino U.M.I.* (7) **2**-A (1989), 345–352.